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On dilute unitary random matrices

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Abstract. We study random dilution of random matrices $H_N = U_N F_N U_N^\dagger$, where U_N are uniformly distributed over the group of $N \times N$ unitary matrices and F_N are non-random Hermitian matrices. We show that the eigenvalue distribution function of dilute random matrices $[H_N]_d$ converges to the semicircle (Wigner) distribution in the limit $N \to \infty$, $p \to \infty$, where p is the dilution parameter. This convergence can be explained by the observation that the dilution eliminates statistical dependence between the entries of $[H_N]_d$. The same statement is valid for the entries of $[U_N]_d$. Our results support the conjecture that the Wigner law is valid for wide classes of dilute Hermitian random matrices.

Random matrices of large dimensions are at present of considerable interest due to applications in various branches of theoretical physics, such as solid-state theory, statistical mechanics (including neural network theory), quantum chaos theory, quantum field theory, and others (see, e.g., monographs and reviews [1–5] and references therein).

Originally large random matrices were used in the middle of the 1950s in statistical nuclear physics, where they were proposed to model energy levels of heavy atomic nuclei [6, 7]. Such nuclei consist of a large number ($N \sim 100$) of particles interacting with each other. Therefore, it was natural to consider the eigenvalues of $N \times N$ symmetric (or Hermitian) matrices A_N whose entries are of the same order of magnitude. In a statistical approach these entries are assumed to be independent identically distributed (i.i.d.) random variables.

The semicircle (or Wigner) law can be regarded as a primary result in the spectral theory of random matrices. It concerns the asymptotic behaviour as $N \to \infty$ of the normalized eigenvalue counting function of symmetric A_N

$$\sigma(\lambda; A_N) = \#\{\lambda_j^{(N)} \leqslant \lambda\} N^{-1}$$

where $\lambda_j^{(N)}$ are eigenvalues of A_N. It was proved in [6] that if

$$A_N(x, y) = \frac{1}{\sqrt{N}} a(x, y) \qquad x, y = \overline{1, N} \tag{1}$$

where $a(x, y)$, $x \leqslant y$, are independent random variables with zero average, variance v^2 and all other moments finite, then $\sigma(\lambda; A_N)$ weakly converges as $N \to \infty$ to a non-random function $\sigma_{\mathrm{sc}}(\lambda; v^2)$ with derivative of semicircle form:

$$\sigma'_{\mathrm{sc}}(\lambda; v^2) = \frac{1}{2\pi v^2} \begin{cases} \sqrt{4v^2 - \lambda^2} & \text{if } |\lambda| \leqslant 2v \\ 0 & \text{if } |\lambda| > 2v. \end{cases} \tag{2}$$

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In modern theoretical physics applications, random matrices with dependent entries have attracted more and more attention. An important example is given by the ensemble of random unitary matrices U_N that have uniform distribution over the unitary group of N dimensions. This ensemble, known as the circular unitary ensemble (CUE), was considered first by Dyson [8] in the early 1960s. At present various random matrix ensembles related with the CUE are widely used in models of quantum transport in mesoscopic systems, quantum chaos theory and other fields (see [4, 9] and references therein). The spectral and related properties of the CUE and other circular ensembles have been extensively investigated [1, 10–13].

In the present paper we study the eigenvalue distribution of dilute versions of random matrices that are constructed using U_N. Random dilution $[\cdot]_d$ of a matrix A_N means that $[A_N]_d$ has, on average, p non-zero entries per row. Such matrices can provide an improved physical description of large systems, where some interactions between elements are broken (see, e.g., [14–16]).

The eigenvalue distribution of symmetric dilute random matrices $[A_N]_d$ with independent entries was studied in [16–18]. It was shown that if

$$[A_N]_d(x, y) = \frac{1}{\sqrt{p}} \begin{cases} a(x, y) & \text{with probability } p/N \\ 0 & \text{with probability } 1 - pN \end{cases} \qquad x \leqslant y$$

where $a(x, y)$ are as in (1), then in the limit $p, N \to \infty$, $p = o(n)$, the function $\sigma(\lambda; [A_N]_d)$ converges to $\sigma_{\rm sc}(\lambda; v^2)$.

The dilution of random matrices with weakly-dependent entries was considered in [19]. The weak dependence means that the matrix elements become independent when spaced widely enough. It was shown that the limiting as $N, p \to \infty$ eigenvalue distribution function of such matrices is again the semicircle one. This was explained by the observation that random dilution eliminates the weak dependence between matrix elements.

In this paper we study the dilution of random matrices that in the pure (undiluted) case have the form $H_N = U_N F_N U_N^\dagger$. These matrices also have dependent entries but the correlations do not decay when the distance between entries increases. However, we show that in this case dilute random matrices $[H_N]_d$ again obey the Wigner law.

To define random unitary matrices U_N, let us consider the group $\mathcal{U}(N)$ of unitary $N \times N$ matrices and introduce the invariant (Haar) measure $\mathrm{d}U_N$ on $\mathcal{U}_N$. We normalize this measure to unity such that $(\mathcal{U}(N), \mathrm{d}U_N)$ can be regarded as the probability space. We denote by $\langle\cdot\rangle_u$ the mathematical expectation with respect to this measure.

Our main result is given by the following statement.

Theorem 1. Let $d_N(x, y)$, $x \leqslant y$, be independent random variables (also independent from U_N) such that

$$d_N(x, y) = \frac{1 - \delta(x - y)}{\sqrt{p}} \begin{cases} 1 & \text{with probability } p/N \\ 0 & \text{with probability } 1 - pN \end{cases}$$

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \tag{3}$$

and let

$$[H_N]_d(x, y) = \sqrt{N}(U_N F_N U_N^\dagger)(x, y) d_N(x, y) \qquad d_N(x, y) = d_N(y, x) \tag{4}$$

where F_N is a non-random $N \times N$ Hermitian matrix. If $\sup_N \|F_N\|$ is bounded and if there exists a finite limit

$$f_2 = \lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} F_N^2 \tag{5}$$

then the measure $\mathrm{d}\sigma(\lambda; [H_N]_d)$ weakly converges in probability as $N, p \to \infty$ and $p = o(N)$ to the semicircle distribution $\mathrm{d}\sigma_{\mathrm{sc}}(\lambda; f_2)$.

Remarks.

(1) We put the factor $1 - \delta(x - y)$ in (3) because the diagonal elements $\sqrt{N}(U_N F_N U_N^\dagger)(x, x)$ have non-zero average of order $\sqrt{N}$. This can cause the divergence of the moments of H_N in the limit $N \to \infty$. One could replace $1 - \delta(x - y)$ by 1, if the condition $N^{-1} \operatorname{Tr} F_N \to 0$ is added to (5). This would not alter the statement of theorem 1.

(2) Random variables $d_N(x, y)$ are small in a certain sense and the factor $\sqrt{N}$ in (4) stands to compensate this. As we shall see from the proof of theorem 1, one can consider, instead of $\sqrt{N} d_N(x, y)$, i.i.d. random variables $a(x, y), x \leqslant y$, defined in (1). Theorem 1 remains true for the ensemble of random matrices

$$[H_N]_a(x, y) = (U_N F_N U_N^\dagger)(x, y) a(x, y) \tag{6}$$

with f_2 changed by $f_2 v^2$.

(3) Under the weak convergence of measures $\mathrm{d}\sigma(\lambda; [H_N]_d)$ we mean that the random variables $\int \varphi(\lambda)\, \mathrm{d}\sigma(\lambda; [H_N]_d)$ converge in probability to $\int \varphi(\lambda)\, \mathrm{d}\sigma_{\mathrm{sc}}(\lambda; f_2)$ for each fixed $\varphi \in \mathbb{C}_0^\infty \mathbb{R}$.

(4) The appearance of the semicircle distribution can be explained by the fact that non-zero entries of $[H_N]_d$ become uncorrelated in the limit $N \to \infty$. We discuss this at the end of the paper.

Proof. We study the moments

$$M_j^{(N)} = \langle\langle L_j^{(N)} \rangle_u \rangle_d \qquad L_j^{(N)} = \frac{1}{N} \operatorname{Tr} [H_N]_d^j = \int \lambda^j \, \mathrm{d}\sigma(\lambda; [H_N]_d)$$

where $\langle \cdot \rangle_d$ denotes the mathematical expectation with respect to the measure generated by $\{d_n(x, y)\}$. We are going to show that for fixed j

$$\lim_{p,N\to\infty, p=o(N)} M_j^{(N)} = \bar{M}_j \qquad \bar{M}_j = \begin{cases} f_2^k \dfrac{(2k)!}{k!(k+1)!} & \text{if } j = 2k \\ 0 & \text{if } j = 2k+1 \end{cases} \tag{7}$$

and

$$\lim_{p,N\to\infty, p=o(N)} [\langle\langle L_j^{(N)} L_j^{(N)} \rangle_u \rangle_d - \langle\langle L_j^{(N)} \rangle_u \rangle_d \langle\langle L_j^{(N)} \rangle_u \rangle_d] = 0. \tag{8}$$

It is known [20] that the moments $\bar{M}_j$, $j = 1, 2, \ldots$, uniquely define a measure $\mathrm{d}\sigma$ such that $\bar{M}_j = \int \lambda^j \, \mathrm{d}\sigma(\lambda)$. In [6] it was proved that this measure is the semicircle distribution (2),

$$\bar{M}_j = \int \lambda^j \, \mathrm{d}\sigma_{\mathrm{sc}}(\lambda; f_2).$$

Using (7) and (8), it is easy to derive the weak convergence in probability of the measures $\mathrm{d}\sigma(\lambda; [H_N]_d)$. Indeed, one can consider functions $f_N(z) = \int (\lambda - z)^{-1} \, \mathrm{d}\sigma(\lambda; [H_N]_d) z \in$

$\mathbb{C}\backslash\mathbb{R}$ and derive from (7) and (8) the convergence in probability of $f_N(z)$. This can be easily done with the help of the representation

$$f_N(z) = -\sum_{j=0}^{2m} L_j^{(N)} z^{-p-1} - R_N^{(m)}(z) z^{-2m-1} \qquad |R_N^{(m)}(z)| \leqslant L_{2m}^{(N)}(1 + |\mathrm{Re}\, z(\mathrm{Im}\, z)^{-1}|).$$

Since the functions $(\lambda - z)^{-1}$ are everywhere dense in $\mathbb{C}_0^\infty(\mathbb{R})$, then the convergence mentioned in remark 3 is shown.

We start with the proof of (7) and rewrite $M_j^{(N)}$ in the form

$$M_j^{(N)} = \frac{1}{N} \sum_{x,\bar{y}} \sum_{\bar{s},\bar{t}} \langle \Phi_j(x, \bar{y}, \bar{s}, \bar{t})\rangle_u \langle \Psi_j(x, \bar{y})\rangle_d \tag{9}$$

where $\bar{y} = \{y_1, y_2, \ldots, y_{j-1}\}$, $\bar{s} = \{s_1, s_2, \ldots, s_j\}$ and $\bar{t} = \{t_1, t_2, \ldots, t_j\}$

$$\Phi_j(x, \bar{y}, \bar{s}, \bar{t}) = U(x, s_1) F(s_1, t_1) U^\dagger(t_1, y_1) \ldots U(y_{j-1}, s_j) F(s_j, t_j) U^\dagger(t_j, x)$$

and

$$\Psi_j(x, \bar{y}) = N^{j/2} d(x, y_1) d(y_1, y_2) \ldots d(y_{j-1}, x).$$

The last average in (9) is easy to compute according to definition (3). The average $\langle \Phi_j \rangle_u$ can be found with the help of the following statement proved in [21, 22] and summarized in [9].

Proposition 1. Let a_i, b_i and $\alpha_{i'}$, $\beta_{i'}$ be the sets of fixed numbers. Then

$$\langle U(a_1, b_1) \ldots U(a_q, b_q) \overline{U(\alpha_1, \beta_1)} \ldots \overline{U(\alpha_r, \beta_r)} \rangle_u = \delta_{qr} \sum_{P,P'} V_{c_1,\ldots,c_n} \prod_{i=1}^{r} \delta_{a_i \alpha_{P(i)}} \delta_{b_i \beta_{P'(i)}} \tag{10}$$

where the overline means the complex conjugate and the summation runs over all permutations P and P' of the numbers $1, \ldots, r$. The coefficient V depends on the set of cyclic permutations $(c_1, \ldots, c_n)$, $\sum_l^n |c_l| = r$, that determine the unique factorization $P^{-1}P' = c_1 \ldots c_n$. The leading term of V is given by the formula

$$V_{c_1,\ldots,c_n} = \prod_{l=1}^{n} V_{c_l} + O(N^{n-2r-2}) \tag{11a}$$

where

$$V_c = N^{1-2c} \frac{(-1)^{c-1}}{c} \binom{2c-2}{c-1} + O(N^{-1-2c}). \tag{11b}$$

In [9] relations (10) and (11) were used to compute averages of the type

$$\mathrm{Tr}\,(F_1 U \ldots F_q U G_1 U^\dagger \ldots G_r U^\dagger).$$

To do this, a diagram technique was developed. Here we modify the technique suggested in [9] to study averages (9) in the limit $N, p \to \infty$.

The diagrams consist of elements shown in figure 1. We denote $U(x, s)$ by a thin dotted arrow. This line starts at the black ball and ends at the white ball. These balls denote variables x and s, respectively. Element $U^\dagger(t, y)$ is represented by a thin dotted line arrow that starts at the white ball and ends at the black one. $F(s, t)$ is given by a thick line joining two white balls and $d(x, y)$ is given by a thick dotted line joining two black balls. A thin line joining two white or two black balls means that corresponding variables are equal (take equal values).

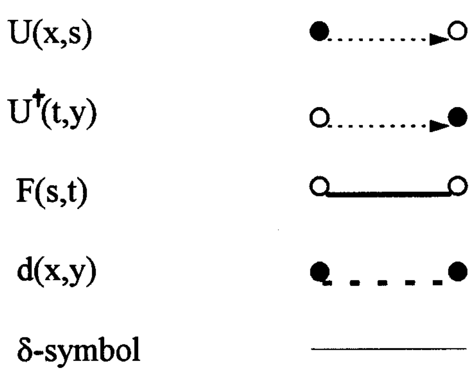

Figure 1. Diagram denotations of matrices U_N, $U_N^{\dagger}$, F_N, d_N, and the Kronecker δ-symbol.

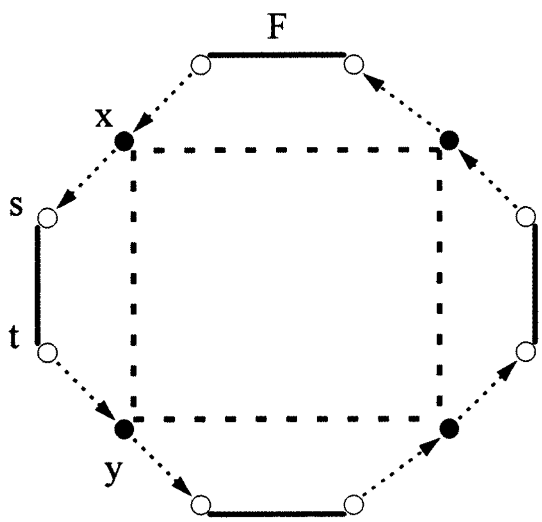

Figure 2. A diagrammatic representation of the term $\Phi_6\Psi_6$.

Let us first consider the case of $j = 2k$. Then the product $\Phi_{2k}(x, \bar{y}, \bar{s}, \bar{t})\Psi_{2k}(x, \bar{y})$ for fixed $x, \bar{y}, \bar{s}, \bar{t}$ can be presented by a close circuit formed by the lines with arrows and thick lines joining white and black balls. We denote such a diagram by γ_{2k}. The case of $k = 3$ is given in figure 2.

Suppose for a moment that the thick dotted lines ending at x are absent in γ_{2k}. Then we can sum over x and then over $s_1 = s_{2k}$. Apparently, if all thick dotted lines are absent, we obtain $N^{-1}\,\mathrm{Tr}\,F^{2k}$ for (9).

Let us explain briefly why the presence of thick dotted lines changes the result. Broadly speaking, the reason is that the average over random variables d is non-zero only in the cases when the black balls in γ_{2k} are paired or, in other words, glued. Our main observation is that the leading contribution to (9) in the limit $N, p \to \infty$ is provided by those diagrams that consist of blocks presented in figure 3(A). Proposition 1 implies that the average over $\mathrm{d}U_N$ can be performed as if these blocks are factorized, i.e. considered as the separate k diagrams. This produces the factor $(N^{-1}\mathrm{Tr}F_N)^k$.

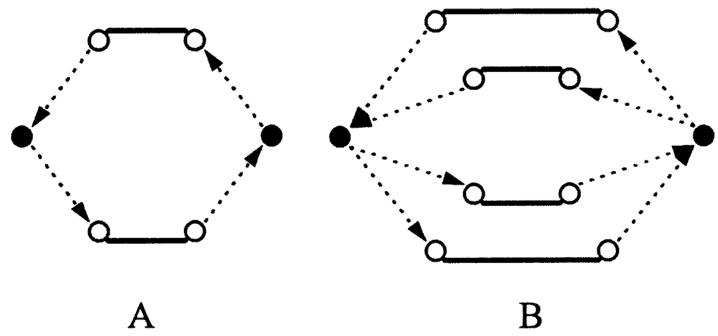

Figure 3. Single (A) and multiple (B) blocks to construct diagrams δ_{2k}.

Thus, we split the rigorous proof of (7) into two steps. In the first stage we perform the average over random variables d and separate those diagrams that provide a non-zero contribution to (7). Then, in the second stage, we examine the average over $\mathrm{d}U_N$ of (9) to obtain the expression for the leading terms.

We study the average over d with the help of the procedure developed in the main by Wigner [6]. Let us regard a fixed sequence $(x \equiv y_0, y_1, \ldots, y_{j-1}, y_j \equiv x)$ as a 'walk' consisting of $2k$ steps. The value $\langle \Psi(x, Y) \rangle_d$ depends on the number of steps that have no inverse and the number of steps that have an inverse. We will say that a step and its inverse make a pair. We rearrange summation over $\bar{y} = \{y_1, y_2, \ldots, y_{j-1}\}$ in the following way: we indicate steps that are paired and then allow variables $y_1, y_2, \ldots, y_{j-1}$ to move but in a way that conserves this pairing. Then we sum the contributions over all possible pairings (including the case of no pairs). We are going to show that the leading contribution comes from the set of pairings where walks are such that each step has its inverse. We denote this set of pairings by Y_{2k}.

If pairs of steps are indicated, then in γ_{2k} a pair of thick dotted lines is pointed out. Black balls that belong to these lines correspond to variables y_i that take equal values. We remove these two thick dotted lines from γ_{2k} and glue up two pairs of black balls. We repeat this procedure until all thick dotted lines are removed. As a result, we obtain from γ_{2k} a new diagram δ_{2k}.

Each pairing from Y_{2k} produces a new diagram, so we obtain the set Δ_{2k} of diagrams δ_{2k}. Each δ_{2k} is constructed by k blocks of the form given in figure 3. These blocks are glued in black ball points. An example of δ_{18} is given in figure 4.

In this diagram there are seven blocks that we call 'single' (i.e. such that they have the form given in figure 3) and one block that we call 'multiple' (or m-fold). Such a block is constructed from m single blocks that are inserted one to another (in figure 4, m is equal to 2). We also refer to the single or multiple blocks that have one black ball unglued as 'free'. There are two free blocks in figure 4. There is also a closed chain consisting of four single blocks.

Now we turn to the second stage of the proof of (7). We are going to show that, in the limit $N, p \to \infty$, the non-zero contribution comes only from the diagrams that consist of elementary single blocks. If this is proved, the result (7) is easy to derive, because the number of such diagrams is $(2k)!/[k!(k+1)!]$ [6] and, according to proposition 1, each of them provides the leading contribution f_2.

To perform averaging over $\mathrm{d}U_N$ and summation over $x, \bar{y}, \bar{s}, \bar{t}$ for each particular $\delta_{2k} \subset \Delta_{2k}$, we formulate the following rules based on proposition 1:

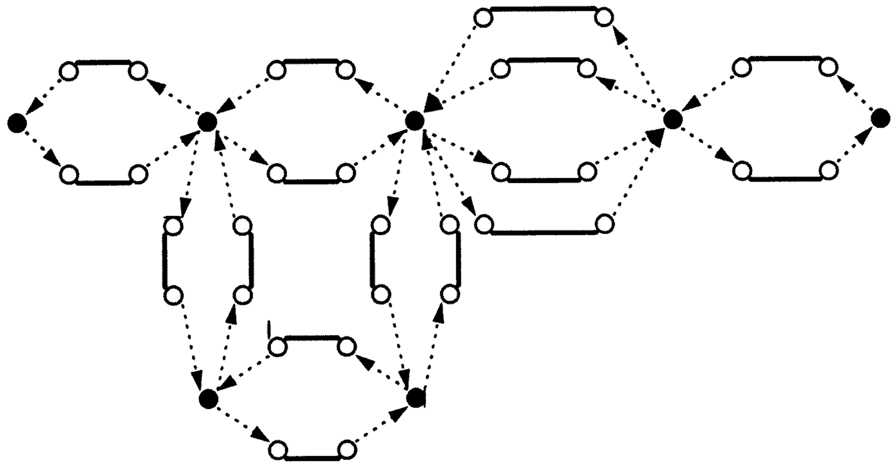

Figure 4. An example of a diagram $\delta_{2k} \subset \Delta_{2k}$ for $k = 9$.

(a) join white balls by thin lines; each ball is to be joined only with one other ball;

(b) find all closed circuits formed by thin lines and thin dotted arrows; each such circuit, called a U-cycle, provides a factor V_q, where q is half of the cycle length (i.e. half of the number of dotted arrows involved);

(c) find all closed circuits formed by thin and thick lines; each such circuit, called an F-cycle, provides a factor $\mathrm{Tr}\, F^r$, where r is a number of thick lines involved;

(d) count the number z of black balls in δ_{2k}; they give the factor $N(N-1)(N-2)\ldots(N-z+1) = N^z(1+O(1))$; finally, multiply the contribution by a factor N^{-1} standing in front of the average (9).

Let us note that the summation over y_i is such that there is no coincidence between variables corresponding to different black balls. Then the random variables d_N from different blocks are jointly independent.

Let us denote by $\Pi(\delta_{2k})$ the set of diagrams π_{2k} obtained from δ_{2k} by drawing thin lines. It is clear that diagrams π_{2k} provide terms of different orders.

It follows from (11) that $V_q = o(V_1^q)$. It is apparent that $\mathrm{Tr} F^{r+s} = o(\mathrm{Tr}\, F^r\, \mathrm{Tr}\, F^s)$. Thus, the leading contribution provided by δ_{2k} comes from those diagrams π_{2k}, where the number of U-cycles of length 2 and F-cycles of length 2 is maximal. We call such cycles 'elementary'.

This condition implies that thin lines are drawn within each single or multiple block and they join those white balls that are attached by dotted lines to the same black ball. Indeed, if one draws a thin line joining balls from two different blocks, then either V_q with $q > 1$ or F^r with $r > 2$ will arise.

In a single block, there is only one possibility to produce elementary cycles. For an m-fold block, there are $m!$ possibilities (see figures 5(A) and 5(B), respectively). Let us denote by $\Pi^*_{2k} \subset \Pi_{2k}$ the subset that consists of diagrams π^*_{2k} providing the leading contribution.

Now we can compute the leading contribution from a diagram π^*_{2k}. Let us start reading from free single blocks. The pair $[\sqrt{N} d_N]^2$ is independent from the rest of variables d_N and this provides the factor $\langle [\sqrt{N} d_N]^2 \rangle_d = 1$.

The sum over the free moving variable that corresponds to the black ball which is not glued is normalized by the elementary U-cycle involving this ball. The F-cycle is normalized

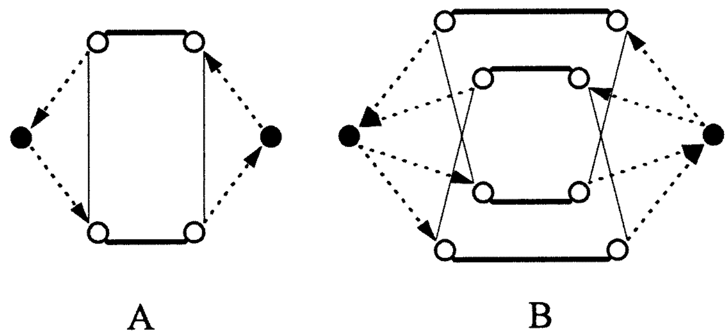

Figure 5. Elementary cycles for single (A) and multiple (B) blocks.

by another elementary U-cycle from this block. Thus, each free single block provides the factor $N^{-1}\,\mathrm{Tr}\,F_N^2 = f_2(1+o(1))$ to the contribution.

If there are l free single blocks, then we can reduce the diagram π^*_{2k} to a diagram π^*_{2k-2l} by removing l free blocks and multiplying the contribution by f_2^l.

Let us compute the factor that comes from the multiple (m-fold) block. There are m coinciding pairs of random variables $\sqrt{N}d_N$ that are independent from other pairs. This gives the factor $\langle[\sqrt{N}d_N]^{2m}\rangle_d = N^{m-1}p^{1-m}$. There are $2m$ elementary U-cycles and m elementary F-cycles. The free black ball provides the factor N. Gathering these factors, we obtain that $m!$ possible drawings provide the leading contribution $m!\,f_2^m p^{1-m}(1+o(1))$ as $N \to \infty$. We see that multiple blocks are responsible for $1/p$-corrections to the result and the diagrams containing multiple blocks provide a vanishing contribution to the average (9) in the limit $N, p \to \infty$.

This shows that π^*_{2k} obey a further reduction when free multiple blocks are taken into account.

It is easy to see that at the end of these steps of reduction, one arrives either at a solitary block (single or multiple) or at a closed chain (or several closed chains glued in black ball points). In the first case the single block possesses two elementary U-cycles, one elementary F-cycle and two free black balls. Remembering the factor N^{-1} from (9), we come again to the factor $f_2(1+o(1))$. Apparently, an m-fold block provides a factor $O(p^{1-m})$.

Let us compute the contribution from a closed circuit constructed from l single blocks. There are l black balls, $2l$ elementary U-cycles and l elementary F-cycles. Regarding the factor N^{-1}, we obtain that such a closed chain provides a factor $O(N^{-1})$ to the result.

It is clear that if the closed circuit involves multiple blocks or if several closed chains are glued in black ball points, then the contribution is $o(N^{-1})$ in the limit $N, p \to \infty$.

Summing up previous considerations, we conclude that the contribution of order $O(1)$ comes from the set $\bar{\Delta}_{2k}$ of diagrams $\bar{\delta}_{2k}$ that are constructed from single blocks and have no closed chains. The corresponding diagram π^*_{2k} is unique and provides the factor $[f_2]^k(1+o(1))$.

Turning back to the diagram γ_{2k} as displayed in figure 2, we observe that each diagram from Δ_{2k} is determined by the splitting of the set of thick dotted lines into k pairs such that there is no coincidence between pairs.

The situation is similar to that when the averages $N^{-1}\,\mathrm{Tr}\,D_N^{2k}$ or $N^{-1}\,\mathrm{Tr}\,W_N^{2k}$ are considered and the leading term has been required. It is known [6] that the set $\bar{Y}_{2k}$ of walks, where each step is paired and there is no coincidence between pairs, consists of $\nu_k = (2k)!/[k!(k+1)!]$ elements.

To complete the proof of the first equality of (7), it remains to show that summation over pairings $\hat{Y}_{2k}$, where at least one step has no inverse, provides a vanishing contribution to (7).

Since the walk $(x, y_1, \ldots, y_{j-1}, x)$ starts and ends at the same point, there are at least three more steps that have no inverse. Suppose that the rest of the walk belongs to $\bar{Y}_{2k-4}$. Then this part of the diagram can be reduced and we obtain a closed circuit as given in figure 2 with $2k = 4$ but with no thick dotted lines. The latter means that random variables d_N are independent and we obtain the factor $\langle[\sqrt{N}d_N]\rangle_d^4 = p^2N^{-2}$. Now we can perform the sum over the corresponding y_i and the only restriction is that these variables take different values.

However, we allow them to take all values and this leads to additive corrections of order $o(1)$ to the results. This is because the expression $(UFU^\dagger)(x, y_1)\ldots(UFU^\dagger)(y_{j-1}, x)$ is bounded, no matter whether y_i are fixed or not.

Therefore, we can sum over y_i as is shown in figure 2 and obtain the factor $N^{-1}\,\mathrm{Tr}\,F_N^4$, where N^{-1} comes from (9). Now it is clear that summation over pairings $\hat{Y}_{2k}$ provides a vanishing contribution as $N, p \to \infty$.

To complete the proof of (7), it remains to show that the odd moments of H_N vanish. This is easy to see, observing that the arbitrary walk $(x, y_1, \ldots, y_{2k-1}, y_{2k}, x)$ has at least three different steps that have no pairs. If the rest of the diagram γ_{2k+1} provides the factor $O(1)$, then according to the previous argument it can be reduced. We come to the blocks that provide a factor $\langle\sqrt{N}d_N\rangle_d^3 = (\sqrt{p}/\sqrt{N})^3$.

Thus, (7) is proved.

Let us note that if one avoids the condition $d_N(x, x) = 0$, then the blocks formed by two arrows and one thick line can appear in the diagram π_{2k}^*. However, such blocks provide factors $N^{-1}\,\mathrm{Tr}\,F_N$ that vanish due to the condition mentioned in remark 1.

Let us briefly describe the proof of (8) that reflects the self-averaging property of the measure $\mathrm{d}\sigma(\lambda; [H_N]_d)$. According to definition of $L_j^{(N)}$, we have to show that the variable

$$S_{2k}^{(N)} = \frac{1}{N^2}\sum_{x,x'}\sum_{\bar{y},\bar{s},\bar{t}}\sum_{\bar{y}',\bar{s}',\bar{t}'}[\langle\Phi_{2k}\Phi'_{2k}\rangle_u\langle\Psi_{2k}\Psi'_{2k}\rangle_d - \langle\Phi_{2k}\rangle_u\langle\Phi'_{2k}\rangle_u\langle\Psi_{2k}\rangle_d\langle\Psi'_{2k}\rangle_d] \tag{12}$$

is of order $o(1)$ as $N, p \to \infty$. The terms in the square brackets can be both represented (prior to averaging) by the same diagram $\gamma_{2k}\cup\gamma'_{2k}$, where γ_{2k} and γ'_{2k} are as in figure 3.

Let us first consider summation over those pairings where

$$\langle\Psi_{2k}\Psi'_{2k}\rangle_d = \langle\Psi_{2k}\rangle_d\langle\Psi'_{2k}\rangle_d. \tag{13}$$

Apparently, we can restrict ourself to the sum over $\bar{Y}_{2k}$ and $\bar{Y}'_{2k}$ providing leading contribution. Equality (13) means that pairs determined by the sum over y_i and pairs determined by the sum over y'_i are different and represent independent random variables. This reduces the diagram $\gamma_{2k}\cup\gamma'_{2k}$ to the diagram $\delta_{2k}\cup\delta'_{2k}$ for both terms in square brackets of (12). According to rules (a)–(c), the leading contributions from both terms are equal and, being subtracted, provide a vanishing contribution to (12).

Let us turn to the case when

$$\langle\Psi_{2k}\Psi'_{2k}\rangle_d \neq \langle\Psi_{2k}\rangle_d\langle\Psi'_{2k}\rangle_d. \tag{14}$$

There are two different cases to consider:

(i) summation goes over walks belonging to $\bar{Y}_{2k}$ and $\bar{Y}'_{2k}$; and

(ii) one of the two walks or both of them belong to $\hat{Y}_{2k}$ or $\hat{Y}'_{2k}$, respectively.

Let us consider case (i). Relation (14) means that at least one pair from $\bar{Y}_{2k}$ has a repetition in $\bar{Y}'_{2k}$. Therefore, the diagram $\delta_{2k}\cup\delta'_{2k}$ is transformed into a new diagram δ_{4k}

and there is at least one m-fold block in the latter. Taking into account the factor N^{-2} from (12), we easily come to the conclusion that the average over dU of δ_{4k} provides contributions of order $O(N^{-1}p^{-1})$ and of order $O(N^{-2})$ for the first and the second terms from the square brackets of (12), respectively.

Let us consider summation (ii). In all three possibilities described, the second term of (12) provides a vanishing contribution and we can study just the first term. Relation (14) means that some steps from $(x, \bar{y}, x)$ coincide with steps from $(x', \bar{y}', x')$. Then $\delta_{2k} \cup \delta'_{2k}$ is also transformed into one diagram that we denote by $\hat{\delta}_{4k}$. Since there is a factor N^{-2} in (12), then the contribution from averaging over dU and summing over $\bar{y}$, $\bar{y}'$ provides a contribution N^{-1}. Indeed, positive powers N^{β} can occur only due to moments $\langle[\sqrt{N}d_N]^l\rangle_d$, $l \geqslant 2$, that correspond to the multiple blocks in $\hat{\delta}_{2k}$. However, these powers are compensated by elementary U-cycles produced by thin lines inside these blocks. Thus, all terms described in (ii) provide a contribution of order $O(N^{-1})$.

Thus, (8) is shown and theorem 1 is proved. □

To discuss this result, let us note that formulae (10) and (11) imply that relations

$$\langle|U_N(x, y)|^2|U_N(x, z)|^2\rangle_u - \langle|U_N(x, y)|^2\rangle_u\langle|U_N(x, z)|^2\rangle_u = -\frac{1}{2N^3}(1 + o(1)) \tag{15}$$

and

$$\langle|H_N(x, y)|^2|H_N(x, z)|^2\rangle_u - \langle|H_N(x, y)|^2\rangle_u\langle|H_N(x, z)|^2\rangle_u = -\frac{1}{2N^3}f_2^2(1 + o(1)) \tag{16}$$

hold provided $x \neq y$, $x \neq z$, and $y \neq z$. These equalities imply that for fixed N the entries of the random matrices U_N and H_N are correlated and these correlations do not decay when the distance between entries increases.

Thus, the appearance of the Wigner distribution in dilute matrices (4) is provided by a mechanism that is different to the one described in [19] for random matrices with weakly-dependent entries.

Indeed, relations (15) and (16) mean that correlations between entries of $\sqrt{N}U_N$ and $\sqrt{N}H_N$ (cf (1)) are characterized by values of order $O(N^{-1})$ and hence vanish in the limit $N \to \infty$. Our computations of the proof of theorem 1 show that the limiting transition $N, p \to \infty$ can be performed subsequently: first $N \to \infty$ and then $p \to \infty$. Thus, for a fixed value of p, the asymptotics of large N lead to a dilute matrix with independent entries. Such a matrix belongs to the class studied in [16–18], where the Wigner law is proved to be valid in the limit $N, p \to \infty$.

In the case of random modulation of $H_N(x, y)$ by i.i.d. $a(x, y)$ (6), matrix elements $[H_N]_a(x, y), x \leqslant y$, become uncorrelated random variables. Our result concerning $\sigma(\lambda; [H_N]_a)$ means that in the Wigner law the absence of correlation between matrix entries plays a more important role than the independence property.

Summing up, we can deduce that the Wigner law in dilute random matrices (4) is valid because random dilution eliminates the dependence between entries of random matrices $U_N F_N U_N^{\dagger}$ in the limit $N \to \infty$. To conclude, let us note that a similar elimination holds for matrix elements $[U_N]_d$. Using our technique, one can easily prove the following statement.

Theorem 2. Let unitary random matrices U_N be as in theorem 1 and

$$[U_N]_d(x, y) = \sqrt{N}U_N(x, y)d_N(x, y)$$

where $d_N(x, y)$, $x \leqslant y$, are independent random variables given by (3). Let us consider random operators $H_N^{(d)}$ with the entries

$$H_N^{(d)}(x, y) = ([U_N]_d F_N[U_N^{\dagger}]_d)(x, y).$$

If there exists

$$\varphi(\lambda) = \lim_{N\to\infty} \sigma(\lambda; F_N)$$

then $d\sigma(\lambda; H_N^{(d)})$ converges as $N, p \to \infty$, $p = o(N)$, to a non-random measure $d\psi(\lambda)$. The Stieltjes transform $f(z) = \int(\lambda - z)^{-1}\, d\psi(\lambda)$, $\operatorname{Im} z \neq 0$ can be found from

$$f(z) = \left[-z + \int_{-\infty}^{\infty} \frac{\tau\, d\varphi(\tau)}{1 + \tau f(z)} \right]^{-1}. \tag{17}$$

Remark. The difference between the limiting eigenvalue distributions of $H_N = U_N F_N U_N^\dagger$ and $H_N^{(d)}$ becomes especially clear when $F_N \equiv v^2 I$. Obviously, for the first ensemble we have

$$\sigma(\lambda; H_N) = \chi_{-\infty, v^2}(\lambda) \equiv \begin{cases} 0 & \text{if } \lambda < v^2 \\ 1 & \text{if } \lambda \geqslant v^2. \end{cases}$$

In the second case, one can easily obtain an explicit form for $\psi(\lambda)$,

$$\psi(\lambda) = \int_{-\infty}^{\lambda} (2\pi v^2 \sqrt{\mu})^{-1} \sqrt{4v^2 - \mu}\, d\mu. \tag{18}$$

This can be derived from (17) with $\varphi(\lambda) = \chi_{-\infty, v^2}(\lambda)$ and the inversion formula for the Stieltjes transform [23]:

$$\psi(b) - \psi(a) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \int_a^b \operatorname{Im} f(\lambda + \iota\varepsilon)\, d\lambda$$

where a and b are the continuity points of $\psi(\lambda)$. It is not hard to see that distribution (18) is related to the semicircle one (2),

$$\psi(\lambda) = \sigma_{\mathrm{sc}}(\sqrt{\lambda}; v^2) - \sigma_{\mathrm{sc}}(-\sqrt{\lambda}; v^2).$$

One can also write that

$$\psi(\lambda) = \lim_{N\to\infty} \sigma(\lambda; [U_N]_d [U_N^\dagger]_d) = \lim_{N\to\infty} \sigma(\lambda; A_N^2).$$

This relation can be regarded as the evidence that correlations between entries of $[U_N]_d$ vanish rather fast in the limit $N, p \to \infty$ and do not contribute to the limiting eigenvalue distribution of $[U_N]_d [U_N^\dagger]_d$.

The proof of theorem 2 is based on the observation that the leading terms of the averages of the moments

$$L_j^{(d,N)} = \frac{1}{N} \operatorname{Tr} (H_N^{(d)})^j$$

in the limit $N, p \to \infty$ coincide with the leading terms of the averages of the moments $N^{-1} \operatorname{Tr} (B_{N,N}^{(d)})^j$, where $B_{N,N}^{(d)} = [\Xi_{N,N}]_d F_N [\Xi_{N,N}^{\mathrm{T}}]_d$ and $\Xi_{N,m}(x, \mu) = N^{-1/2} \xi^\mu(x)$, $x = \overline{1, N}$, $\mu = \overline{1, m}$, with i.i.d. random variables $\xi^\mu(x)$ that have zero average, variance 1, and all other moments finite.

Let us note that the random matrix ensemble

$$B_{N,m} = \Xi_{N,m} F_m \Xi_{N,m}^{\mathrm{T}} \tag{19}$$

with $F_m \equiv I$ is widely used in statistical mechanics of disordered systems [24] and in neural network theory [14], where it is known as the Hopfield model of autoassociative memory.

The spectral properties of (19) and its generalizations were first studied in [25]. In particular, equation (17) can be derived from results of [25], if one considers the random matrices $B_{N,N}$ with diagonal F_N that satisfy conditions of theorem 2.

It should be noted that the entries $B_{N,m}(x, y)$ are uncorrelated, but statistically dependent random variables. This dependence vanishes as $N \to \infty$. In [26], it is shown that the normalized eigenvalue distribution function of $[B_{N,m}]_d$ with $F_m \equiv I$ converges as p, m, $N \to \infty$, $p = o(N)$, $m/N \to c > 0$ to the semicircle distribution (see also [27] for the case of non-symmetric dilution of (19)). Apparently, the Wigner law holds here due to the same elimination of the dependence that is observed in the present paper for the dilute unitary random matrices $[U_N]_d$ and matrices $[H_N]_d$.

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